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A CHARACTERIZATION OF THE COMPLEMENTED TRANSLATION-INVARIANT SUBSPACES OF $H^1(\mathbf{R})$

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ABSTRACT. The purpose of this paper is to characterize the complemented translation-invariant subspaces of $H^1(\mathbf{R})$ in terms of the zero set of the Fourier transform. It is shown that if X is such a subspace then $X = I(A)$ where A is in the ring generated by arithmetic progressions and lacunary sequences and A is ε-separated for some $\varepsilon > 0$. This proves a conjecture of the author and D. Ullrich.

0. INTRODUCTION

This paper is essentially a continuation of [AU] where the problem of characterizing the complemented translation-invariant subspaces of $H^1(\mathbf{R})$ was considered. That paper depended heavily on the papers of Klemes [K] and Alspach and Matheson [AM] where the complemented translation-invariant subspaces of $H^1(\mathbf{T})$, respectively, $L^1(\mathbf{R})$ were characterized. In [AU] it was shown that if A is a closed set in the ring of sets generated by the arithmetic progressions and lacunary sequences and A is ε-separated for some $\varepsilon > 0$ then $I(A)$ is complemented in $H^1(\mathbf{R})$. It was also shown that the ε-separation condition is necessary for complementation. Here we will complete the characterization by showing that the algebraic conditions are also necessary. The overall scheme is similar to that used by Klemes but many technicalities are required to implement the scheme.

In [K] Klemes showed that a complemented translation-invariant subspace of $H^1(\mathbf{T})$ has hull in the ring generated by arithmetic progressions and lacunary sequences. (This is also sufficient for complementation.) Our main result is

Theorem 0.1. *Let $X \neq \{0\}$ be a translation-invariant subspace of $H^1(\mathbf{R})$. Then X is complemented if and only if the hull of X is in the ring generated by arithmetic progressions and lacunary sequences and there is some $\varepsilon > 0$ such that the hull is ε-separated.*

Let us now fix some notational conventions and recall some basic definitions which will be used in the paper. $H^1(\mathbf{R})$ denotes the subspace of $L^1(\mathbf{R})$ of

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functions with Fourier transform equal to zero on $\{t \in \mathbf{R}: t \leq 0\}$ and $H^1(\mathbf{T})$ denotes the subspace of $L^1(\mathbf{T})$ of functions with Fourier transform equal to zero on $\{n \in \mathbf{Z}: n < 0\}$. $H_0^1(\mathbf{T})$ denotes the subspace of $H^1(\mathbf{T})$ of functions with Fourier transform vanishing at zero. Let $e_t(s) = e^{its}$ for $s, t \in \mathbf{R}$. The pairing $\langle f, g\rangle$ with $f \in L^\infty(\mathbf{R})$ and $g \in L^1(\mathbf{R})$ will be

$$\langle f, g\rangle = \int f(-s)g(s)\, ds.$$

Thus $\hat{g}(t) = \langle e_t, g\rangle$ and if $f \in L^1(\mathbf{R})$ as well, $f * g(0) = \langle f, g\rangle$. We will say that a subset A of $\mathbf{R}$ is *ε-separated* if $\inf\{|s-u|: s, u \in A, s \neq u\} \geq \varepsilon > 0$. $\Omega_l(\mathbf{R})$ will denote the ring of subsets of $\mathbf{R}$ generated by lacunary sequences (possibly of finite length) and arithmetic progressions. By the term *arithmetic progression* we mean a set of the form $\alpha\mathbf{Z} + \beta$. Note that we do not consider $\mathbf{R}$ itself to be in $\Omega_l(\mathbf{R})$. We will use the notation $I(A)$ for both the subspace of $L^1(\mathbf{R})$ or $L^1(\mathbf{T})$ of functions with transform vanishing on A and its intersection with $H^1(\mathbf{R})$ or $H^1(\mathbf{T})$. If X is a translation-invariant subspace of $L^1(\mathbf{R})$, $Z(X)$ will be the hull of X, i.e., $\{t: \hat{x}(t) = 0 \text{ for all } x \in X\}$. To avoid writing some parentheses we will assume that the algebraic operations take precedence over set theoretic operations, e.g., $B \cup \alpha\mathbf{Z} + \beta = B \cup ((\alpha\mathbf{Z}) + \beta)$. Other standard notation and facts may be found in [K] or [Ru].

1. Preliminaries

It was shown [AU, Theorem 2.3] that the hull of a complemented translation-invariant subspace of $H^1(\mathbf{R})$ is ε-separated for some $\varepsilon > 0$. In particular this implies that such a subspace is $I(A)$ for A equal to the hull of the subspace. In addition it is known that A must be close to a finite union of arithmetic progressions. Precisely

Theorem 1.1 [AU, Theorem 3.5]. *If $I(A)$ is a complemented subspace of $H^1(\mathbf{R})$ then there is a number $\tau > 0$ and a finite set $F \subset [0, \tau)$ such that for every $\delta > 0$, $A \cap [F + \tau\{0, 1, \ldots\} + (-\delta, \delta)]^c$ has arbitrarily large gaps.*

To get the more precise information about A which we require, it is useful to change the problem into a multiplier problem on a different space. Let $b\mathbf{R}$ denote the Bohr compactification of $\mathbf{R}$ and let

$$\mathcal{HM}(b\mathbf{R}) = \{\mu \in \mathcal{M}(b\mathbf{R}): \mu\hat{}\,(t) = 0 \text{ for all } t \leq 0\}.$$

In [AU] it was proved that if $I(A)$ is complemented in $H^1(\mathbf{R})$ then

$$\mathcal{I}(A) = \{\mu \in \mathcal{HM}(b\mathbf{R}): \mu\hat{}\,(t) = 0 \text{ for all } t \in A\}$$

is complemented in $\mathcal{HM}(b\mathbf{R})$. Because $b\mathbf{R}$ is compact this projection may be assumed to be translation-invariant. Because Klemes original argument can be easily modified to apply to multipliers on $\mathcal{HM}(b\mathbf{R})$ we need only consider the case where A fails to have arbitrarily large gaps.

Throughout the paper A will denote a subset of $\mathbf{R}^+$ and m will denote the multiplier on $\mathscr{HM}(b\mathbf{R})$ such that $(m * \mu)^\frown = 1_A \hat{\mu}$, for all $\mu \in \mathscr{HM}(b\mathbf{R})$. Further we will assume that

(a) There is an $\varepsilon > 0$ such that A is ε-separated,

(b) There is a finite set $F \subset \mathbf{R}$ and a positive number τ such that for every $\delta > 0$, $A \setminus [\tau\mathbf{Z} + F + (-\delta, \delta)]$ has arbitrarily large gaps,

(c) There is a number κ such that for any positive real number x, $[x, x+\kappa] \cap A \neq \varnothing$,

(d) For any $\rho > 0$ and $x \in \mathbf{R}$, $A \cap \rho\mathbf{Z} + x$ is a finite union of lacunary sequences. More precisely

$$\sup_y |[y, 2y] \cap A \cap \rho\mathbf{Z} + x| \leq \zeta$$

where ζ depends only on $\|m\|$ and $|\cdot|$ denotes cardinality.

All except (d) have been explained above. For this note that by [AU, Corollary 2.11], $A \cap \rho\mathbf{Z} + x$ must be in $\Omega_l(\rho\mathbf{Z} + x)$ and the proof shows that the corresponding multiplier has norm which depends only on $\|m\|$. By Klemes argument (See (13) of [K].) the norm of the multiplier gives rise to a bound on

$$\sup_y |[y, 2y] \cap A \cap \rho\mathbf{Z} + x|,$$

the parameter which measures the number of lacuna y sequences in a sequence with large gaps. Because of (b) only subsets of $\tau\mathbf{Z}+F$ can fail to have large gaps. However $\tau\mathbf{Z} + F \cap A$ is the support of the transform of a bounded multiplier and therefore can be removed.

Theorem 0.1 will follow immediately from the following result.

Theorem 1.2. *A set A satisfying* (a), (b), (c) *and* (d) *is not the hull of a complemented translation-invariant subspace of* $\mathscr{HM}(b\mathbf{R})$.

In order to simplify the notation and the proof let us note that we can without loss of generality assume that $\tau = 1$ and $F \subset [0, 1)$. Our first task is to get an appropriate family of test measures for use in computing the norm of the multiplier. To construct such measures we use the fact that there is a natural image of $\mathbf{R}$ in $b\mathbf{R}$ and construct measures supported there.

Lemma 1.1. *For every δ, $0 < \delta < 1/2$, there is a positive measure μ on $b\mathbf{R}$ with $\|\mu\| = 1$ such that for all $t \in \mathbf{R}$*

$$\hat{\mu}_\delta(t) = \begin{cases} 1 - |n-t|/\delta & \text{if } |n-t| < \delta \text{ for some } n \in \mathbf{Z}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. There is a positive measure ν on $2\pi\mathbf{Z}$ with transform on $\mathbf{T}$

$$\hat{\nu}(t) = \begin{cases} 1 - |t|/\delta & \text{if } |t| < \delta, \\ 0 & \text{if } \delta \leq |t| \leq 1/2. \end{cases}$$

namely, the Fejér kernel. (See [Ru, p. 23] for the abstract construction.) Observe that the same measure considered on $2\pi\mathbf{Z} \subset b\mathbf{R}$ has the required transform. Q.E.D.

The basic test function we will use is

$$F_{\lambda,\delta} = (e_{-\lambda}K_{2\lambda} + e_{\lambda}K_{2\lambda}) * \left(2\mu_{2\delta} - \mu_{\delta}\right),$$

where $\lambda > \delta > 0$ and

$$\widehat{K}_{\lambda}(t) = \begin{cases} 1 - |t|/\lambda & \text{if } |t| < \lambda, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $\|F_{\lambda,\delta}\| \le 6$, and that

$$\widehat{F}_{\lambda,\delta}(t) = \begin{cases} 1 & \text{if } t \in [n-\delta, n+\delta] \text{ for some } n \in \mathbf{Z},\ |n| \le \lambda - \delta, \\ 0 & \text{if } |t| \ge 3\lambda \text{ or } t \notin (n-2\delta, n+2\delta) \text{ for all } n \in \mathbf{Z}. \end{cases}$$

The next lemma is essentially the same as [K, 2.4].

Lemma 1.2. *If ν is a positive measure on $b\mathbf{R}$ with support of $\hat{\nu} \subset [-s, s]$ for some $s > 0$, $\|\nu\|_{\mathscr{M}(b\mathbf{R})} = 1$, and $\{a_i\}$ is a sequence of N real numbers such that $a_i - a_{i-1} \ge s$, for all i, then*

$$\left\|\sum_{i=1}^{N} e_{a_i}\nu\right\|_{\mathscr{M}(b\mathbf{R})} \le \sqrt{N}.$$

Proof.

$$\left\|\sum_{i=1}^{N} e_{a_i}\nu\right\|_{\mathscr{M}(b\mathbf{R})} = \left\|\sum_{i=1}^{N} e_{a_i}\right\|_{L^1(\nu)} \le \left\|\sum_{i=1}^{N} e_{a_i}\right\|_{L^2(\nu)} = \sqrt{N}$$

because $\int e_{(a_i - a_j)}\,d\nu = 0$ if $i \ne j$. Q.E.D.

Observe that if l is an integer then

$$(e_l K_{2\lambda}) * \mu_\delta = (e_l K_{2\lambda}) * (e_l \mu_\delta) = e_l (K_{2\lambda} * \mu_\delta).$$

Therefore if $\{a_i\} \subset \mathbf{R}$ and $a_i - a_j \ge 2\lambda$ for all $i \ne j$, then

$$\left\|\sum_{i=1}^{N} e_{a_i} F_{\lambda,\delta}\right\|_{\mathscr{M}(b\mathbf{R})} \le 6\sqrt{N},$$

by applying Lemma 1.2 to the pieces of $F_{\lambda,\delta}$.

In place of the gaps used by Klemes we will use a measure of closeness to arithmetic progressions. The next lemma shows that under the assumptions on A the concentration near an arithmetic progression is low.

Lemma 1.3. *For any $\gamma > 0$ there is a λ_0 such that for any $\lambda \ge \lambda_0$ there is a δ_0 such that for all $\delta \le \delta_0$*

$$|[\rho + \{0, 1, 2, \dots, \lambda - 1\} + (-\delta, \delta)] \cap A| \le \lambda\gamma$$

for all ρ, $0 \le \rho < 1$.

Proof. Choose λ_0 such that $\gamma/\zeta > \log_2 \lambda_0/\lambda_0$ where ζ is the constant from assumption (d) above. Suppose the result is false for some $\lambda \ge \lambda_0$. Then for each $i \in \mathbf{N}$ there is a $\rho_i \in [0, 1)$ such that

$$|[\rho_i + \{0, 1, 2, \dots, \lambda - 1\} + (-1/i, 1/i)] \cap A| > \lambda\gamma.$$

By passing to a subsequence we may assume that the sequence ρ_i converges to ρ. It follows that

$$|[\rho + \{0, 1, 2, \dots, \lambda - 1\} + (-\delta, \delta)] \cap A| > \lambda\gamma$$

for every $\delta > 0$. Because $|A \cap [0, \lambda + 1]|$ is finite,

$$|[\rho + \{0, 1, 2, \dots, \lambda - 1\}] \cap A| \geq \lambda\gamma .$$

On the other hand

$$|[\rho + \{0, 1, 2, \dots, \lambda - 1\}] \cap A| \leq \sum_{j=0}^{\log_2 \lambda} |\rho + \mathbf{Z} \cap [\lambda/2^{j+1}, \lambda/2^j] \cap A| \leq \zeta \log_2 \lambda . \quad \text{Q.E.D.}$$

2. Proof of the theorem

Our proof is similar to Klemes' except that in place of [K, Lemma] which yields the constant $p = p(\|m\|)$ in (6) of [K] we get p which depends on $\|m\|$ and on the maximum of the cardinalities of A and A'. This then complicates the application of the lemma in that we must establish that there are sufficiently many terms to generate an estimate similar to (13) of [K].

Before we prove our lemma let us introduce some notation. Let $\lambda \in \mathbf{N}$, $y \in \mathbf{R}$, and let $\delta \in \mathbf{R}$ such that $1/2\delta \in \mathbf{N}$. Define

$$A_k = A \cap y + \{0, 1, \dots, 2\lambda - 1\} + (k\delta, (k+1)\delta),$$
$$B_k = A \cap y + \{-1, -2, \dots, -2\lambda\} + (k\delta, (k+1)\delta)$$

for $k = -1/\delta, \dots, -1, 0, 1, \dots, 1/\delta - 1$. Let

$$N(y, \lambda, \delta) = \sup\{|A_k \cup A_{k+1}| : k = -1/2\delta, \dots, -1, 0, 1, \dots, 1/2\delta - 1\}$$

and

$$M(y, \lambda, \delta) = \sup\{|B_k \cup B_{k+1}| : k = -1/2\delta, \dots, -1, 0, 1, \dots, 1/2\delta - 1\}.$$

Also let us recall the solution of the Littlewood conjecture [MPS].

Theorem 2.1. *There is a constant $C > 0$ such that if G is a compact group with ordered dual then for any trigonometric polynomial f on G*

$$\|f\|_1 \geq C \sum_{i=1}^{N} |\hat{f}(\gamma_i)|/i$$

where $\{\gamma_1 < \gamma_2 < \cdots < \gamma_N\}$ is the support of $\hat{f}$.

Whenever C is used in this paper it will be the constant appearing in Theorem 2.1.

Main Lemma. *Let* $y \in \mathbf{R}$, $\gamma \in \mathbf{N}$ *such that* $y \geq 2\lambda \geq \kappa$ *and* $\delta \in \mathbf{R}$ *such that* $1/2\delta \in \mathbf{N}$. *Let* m *be a* 0-1 *valued multiplier on* $\mathscr{HM}(b\mathbf{R})$ *with support equal to* A. *Suppose that* $p > 4\exp(6C^{-1}\|m\|)$. *If there exists* $r \in \mathbf{N}$ *such that* $\varepsilon 2^{-r-1} > \delta$ *and*

$$(1+p^{-1})^r N(y, \lambda, \delta) > \sup_{|\rho|<2^{-1}} |A \cap [y+\rho-2^{-1}, y+\rho+2\lambda-2^{-1}]|$$

then

$$M(y, \lambda, \delta) \geq N(y, \lambda, \delta)(1+p^{-1})^{s-1}/(p2^s).$$

Proof. Let $N = N(y, \lambda, \delta)$, $M = M(y, \lambda, \delta)$, and $q = (1+p^{-1})^{r-1}/(p2^r)$. Observe that the result is trivial if $N < q^{-1}$ so we assume that $N \geq q^{-1}$.

Suppose the result is false for some y, λ and δ. The argument proceeds by estimating the norm of the multiplier applied to a sequence of test functions.

Let k be the integer nearest to 0 such that $|A_k \cup A_{k-1}| = N$. Note that $|A_{k+j} \cup A_{k+j-1}| \leq N+1$ and $|B_{k+j} \cup B_{k+j-1}| \leq M+1 \leq qN+1$ for all j with $|j| \leq 1/2\delta$. For notational convenience assume that $k = 0$.

For each integer $n < \log_2(\varepsilon/2\delta)$ let

$$L_n = e_{\lambda+y} F_{\lambda, \delta 2^n}$$

and consider $\|m * L_n\|$. If $n = 0$ and $|A_{-2} \cup A_1| < p^{-1}|A_{-1} \cup A_0| = p^{-1}N$ then

$$|B_{-2} \cup B_{-1} \cup B_0 \cup B_1 \cup A_{-2} \cup A_1| \leq [2qN + 2 + p^{-1}N] = Q_1$$

([ ] denotes the greatest integer function) and

$$m * \hat{L_n}(a) = 1 \quad \text{if } a \in A_{-1} \cup A_0 \setminus (y + (-\delta, 0)).$$

($|A \cap y + (-\delta, 0)|$ is at most one.) Thus even if all of the points in $B_{-2} \cup B_{-1} \cup B_0 \cup B_1 \cup A_{-2} \cup A_1$ precede the points in $A_{-1} \cup A_0$ we get that

$$\begin{aligned} \|m * L_n\| &\geq C \sum_{n=Q_1+2}^{Q_1+N} \frac{1}{n} \geq C \log\left(1 + \frac{N-1}{Q_1}\right) \geq C \log\left(1 + \frac{(N-1)/N}{2q + 2N^{-1} + p^{-1}}\right) \\ &\geq C \log(1 + p/4) > 6\|m\| \end{aligned}$$

(because $N \geq q^{-1}$ and $q < 1/2p$). This contradicts the choice of p so it must be the case that

$$|A_{-2} \cup A_1| \geq p^{-1}N.$$

Inductively assume that for $k = 2, 3, \ldots, n-1$ we have shown that

$$\left|\bigcup_{i=-2^{k-1}}^{2^{k-1}-1} A_i\right| \geq (1+p^{-1}) \left|\bigcup_{i=-2^{k-2}}^{2^{k-2}-1} A_i\right| \geq (1+p^{-1})^{k-1} N.$$

Observe that because $N \geq q^{-1}$, it follows that

$$2^k \leq (1+p^{-1})^{k-1} p^{-1} N \leq p^{-1} \left|\bigcup_{i=-2^{k-1}}^{2^{k-1}-1} A_i\right|.$$

Assume that $n \le r+1$ and consider L_n. If

$$\left|\bigcup_{i=-2^{n-1}}^{2^{n-1}-1} A_i\right| < (1+p^{-1})\left|\bigcup_{i=-2^{n-2}}^{2^{n-2}-1} A_i\right|$$

then

$$\begin{aligned}\left|\bigcup_{i=-2^{n-1}}^{2^{n-1}-1} B_i \cup \bigcup_{i=-2^{n-1}}^{2^{n-1}-1} A_i \setminus \bigcup_{i=-2^{n-2}}^{2^{n-2}-1} A_i\right| \\ &< 2^{n-1}qN + 2^{n-1} + p^{-1}\left|\bigcup_{i=-2^{n-2}}^{2^{n-2}-1} A_i\right| \\ &\le [2^{n-1}q[(1+p^{-1})^{n-2}]^{-1} + p^{-1} + p^{-1}]\left|\bigcup_{i=-2^{n-2}}^{2^{n-2}-1} A_i\right| \\ &\le [2^{n-1-r}(1+p^{-1})^{r-n+1}p^{-1} + 2p^{-1}]\left|\bigcup_{i=-2^{n-2}}^{2^{n-2}-1} A_i\right| = Q_n .\end{aligned}$$

Note that

$$m * \widehat{L_n}(a) = 1 \quad \text{if } a \in \bigcup_{i=-2^{n-2}}^{2^{n-2}-1} A_i \setminus (y + (-2^n\delta, 0)).$$

($A \cap y + (-2^n\delta, 0)$ contains at most 1 point since $2^n\delta < \varepsilon$.) As before even if all the points in

$$\mathscr{B}_{n-1} = \bigcup_{i=-2^{n-1}}^{2^{n-1}-1} B_i \cup \bigcup_{i=-2^{n-1}}^{2^{n-1}-1} A_i \setminus \bigcup_{i=-2^{n-2}}^{2^{n-2}-1} A_i$$

precede the points in

$$\mathscr{A}_{n-1} = \bigcup_{i=-2^{n-2}}^{2^{n-2}-1} A_i$$

we get that

$$\begin{aligned}\|m * L_n\| &\ge \sum_{i=|\mathscr{B}_{n-1}|+2}^{|\mathscr{A}_{n-1}\cup\mathscr{B}_{n-1}|} \frac{1}{n} \ge C\log\left[1 + \frac{|\mathscr{A}_{n-1}|-1}{Q_n}\right] \\ &\ge C\log\left[1 + [2^{n-1-r}(1+p^{-1})^{r-n-1}p^{-1} + 2p^{-1}]^{-1}\frac{|\mathscr{A}_{n-1}|-1}{|\mathscr{A}_{n-1}|}\right] \\ &\ge C\log(1+p/4) > 6\|m\| ,\end{aligned}$$

if $r \ge n-1$. This contradicts the choice of p so it must be the case that

$$\left|\bigcup_{i=-2^{n-1}}^{2^{n-1}-1} A_i\right| \ge (1+p^{-1})\left|\bigcup_{i=-2^{n-2}}^{2^{n-2}-1} A_i\right| \ge (1+p^{-1})^{n-1}N .$$

Finally consider L_{r+1}. The estimate above shows that

$$\left|\bigcup_{i=-2^r}^{2^r-1} A_i\right| \geq (1+p^{-1})^r N > \sup_{|\rho|<2^{-1}} |A \cap [y+\rho-2^{-1}, y+\rho+\lambda-2^{-1}]|.$$

Therefore all of the points in $A \cap [y-2^r\delta, y+\lambda-1+2^r\delta]$ are in the set where $\widehat{L}_{r+1}$ is equal to 1. (The ρ is needed in general because $|A_0 \cup A_{-1}|$ may not be equal to N, and then the resulting set is shifted.) This is a contradiction and the lemma is proved. Q.E.D.

In order to utilize the Main Lemma we must show that we can use it iteratively. The next lemma shows that we can use the Main Lemma a large number of times.

Lemma 2.1. *Let* K, $N \in \mathbf{N}$ *and let* $p > 1$. *Then there exists an* $n = n(K) \in \mathbf{N}$ *and positive integers* $K = K_0 > K_1 > K_2 > \cdots > K_n > 0$ *and* $1 \leq r_0 < r_1 < r_2 < \cdots < r_n$ *such that*

(i) $K_{i+1} \leq K_i(1+p^{-1})^{r_i-1}/(p2^{r_i})$, $i = 0, 1, \ldots, n$,

(ii) $NK < K_i(1+p^{-1})^{r_i}$, $i = 0, 1, 2, \ldots, n$,

and for fixed N *and* p,

(iii) $n \to \infty$ *as* $K \to \infty$.

If n *is also fixed,*

(iv) $\liminf_{K\to\infty} K_n/K = a_n > 0$.

Proof. Let $P = 1 + p^{-1}$. Let l be the smallest integer such that $P^l \geq 2$ let s be the smallest integer such that $P^s > N$, and let m be the smallest integer such that $P^{m-1} > p$.

We claim that (i) and (ii) will be satisfied if

(a) $K_{i+1} = [K_i P^{r_i - lr_i - m}]$, $i = 0, 1, \ldots, n$, and

(b) $r_i = (m+1)\sum_{k=0}^{i-1} l^k + l^i s$, $i = 0, 1, 2, \ldots, n$.

Because $K_i P^{r_i-1}/(p2^{r_i}) \geq K_i P^{r_i-1}/(P^{m-1}P^{lr_i})$, (a) implies (i). To prove (ii) we use induction. For $r_0 = s$, (ii) is obvious.

Now assume (ii) holds for $i-1$, then

$$\begin{aligned} K_i P^{r_i} &= [K_{i-1} P^{r_{i-1} - lr_{i-1} - m}] P^{r_i} \\ &\geq K_{i-1} P^{r_{i-1} - lr_{i-1} - m + r_i} - P^{r_i} \geq KNP^{-r_{i-1} + r_{i-1} - lr_{i-1} - m + r_i} - P^{r_i} \end{aligned}$$

by (ii) for $i-1$. We have that $r_i - lr_{i-1} - m = 1$, thus we need only show that $KNP - P^{r_i} \geq KN$. This inequality is satisfied if $KN \geq pP^{r_i}$.

To complete the proof of the lemma we need only observe that for any integer n we can choose K_i and r_i for $i = 1, 2, \ldots, n$ if $KN \geq pP^{r_n}$ and that the definition of the r_i's depends only on N and p. Q.E.D.

We are now ready for the proof of Theorem 1.2.

Proof. As in the Main Lemma choose $p > 1$ such that $C\log(1+p/4) \geq 6\|m\|$. Fix n and $N > 2\kappa|F|$ and let $a_n = \liminf K_n/K$ from Lemma 2.1. By Lemma

1.3 there is a λ_0 such that for any $\lambda \geq \lambda_0$ there is a δ_0 such that for all $\delta < \delta_0$,

$$|[\rho + \{0, 1, 2, \dots, 2\lambda - 1\} + (-\delta, \delta)] \cap A| \leq \lambda a_n/N$$

for all ρ, $0 \leq \rho < 1$. Choose $K \geq \lambda_0$ as in Lemma 2.1 so that there are at least n integers K_i and $K_n/K > a_n/2$. We are given that for any $\delta > 0$ and $\lambda \in \mathbf{N}$ there is a $y_0 \in \mathbf{N}$ such that

$$A \cap [y_0, y_0 + \lambda] \subset F + \{y_0, y_0 + 1, \dots, y_0 + \lambda\} + (-\delta, \delta).$$

Let $\lambda = \inf\{\lambda \in \mathbf{N}$: for every $\delta > 0$ there exists $y \in \mathbf{N}$ and $\alpha \in F$ such that $|A \cap y + \alpha + \{0, 1, \dots, 2\lambda - 1\} + (-\delta, \delta)| \geq K\}$. Note that $2\lambda \leq \kappa |F| K$ because A satisfies (c). Choose $\delta > 0$ such that

$$(*) \qquad |[\rho + \{0, 1, 2, \dots, 2\lambda - 1\} + (-\delta, \delta)] \cap A| \leq \lambda a_n/N < K_n/2$$

for all ρ, $0 \leq \rho < 1$, and such that

$$\delta < \min\{\varepsilon 2^{-r_n - 1}, \varepsilon/2\}.$$

We claim that there exist positive real numbers $\{y_i : i = 0, 1, \dots, n\}$ such that

$$(1) \qquad y_i \geq y_{i+1} + 2\lambda \quad \text{for } i = 0, 1, 2, \dots, n-1,$$

$$(2) \qquad \text{if } C_i = A \cap y_i + \{0, 1, \dots, 2\lambda - 1\} + (-\delta, \delta), \qquad |C_i| = K_i,$$

$$(3) \qquad \text{if } C_i' = A \cap y_i + \{0, 1, \dots, 2\lambda - 1\} + (-2\delta, 2\delta), \qquad |C_i'| \leq 2K_i,$$

$$(4) \qquad \text{if } D_i = A \cap y_i + \{-1, -2, \dots, -2\lambda\} + (-2\delta, 2\delta), \qquad |D_i| \leq 2K_i,$$

$$(5) \qquad \begin{gathered} \text{if } E_i = A \cap y_i + \{2\lambda, 2\lambda + 1, \dots, 4\lambda\} + (-2\delta, 2\delta), \\ |E_i| \leq 2K_{i-1} + 1 \quad \text{for } i \geq 1. \end{gathered}$$

Indeed, let y_0' be the smallest $z_0 \in \mathbf{R}$ such that

$$|A \cap z_0 + \alpha + \{0, 1, \dots, 2\lambda - 1\} + (-\delta, \delta)| = K.$$

We know that such z_0 exist by the choice of λ. Let $y_0 < y_0' + \delta/2$ such that $|C_0| = K$. By the Main Lemma and $(*)$ there is a $z_1 \leq y_0 - 2\lambda$ such that

$$|A \cap z_1 + \{0, 1, \dots, 2\lambda - 1\} + (-\delta, \delta)| = K_1.$$

Let y_1' be the infimum of such z_1 and $y_1 < y_1' + \delta/2$ such that $|C_1| = K_1$. Again by the Main Lemma and $(*)$ there is a $z_2 \leq y_1 - 2\lambda$ such that

$$|A \cap z_2 + \{0, 1, \dots, 2\lambda - 1\} + (-\delta, \delta)| = K_2.$$

Let y_2' be the infimum of such z_2 and $y_2 < y_2' + \delta/2$ such that $|C_2| = K_2$. Continuing in this way we can find a sequence y_i, $i = 1, 2, \dots, n$, satisfying (1) and (2). We claim that (3), (4), and (5) follow immediately from the choice of the y_i's.

Indeed, if $|C_i'| > 2K_i$ then there is a subinterval I of $(-2\delta, 2\delta)$ of length δ such that

$$|A \cap y_i + \{0, 1, \dots, 2\lambda - 1\} + I| > K_i$$

which implies that

$$|A \cap y_i - 1 + \rho + \{0, 1, \dots, 2\lambda - 1\} + (-\delta, \delta)| \geq K_i,$$

where ρ is the midpoint of I. This contradicts the choice of y_i. The arguments for (4) and (5) are similar.

Let $L_i = e_{\lambda+y_i} F_{\lambda,\delta}$ for $i = 0, 1, 2, \dots, n-1$ and let $L = \sum L_i$. By Lemma 1.2, $\|L\|_1 \leq 6\sqrt{n}$. We will now estimate $\|m * L\|$.

First observe that

$$\operatorname{supp} m * L^\frown \subset \bigcup A \cap y_i + \{-2\lambda, -2\lambda + 1, \dots, 0, 1, \dots, 4\lambda\} + (-2\delta, 2\delta)$$

and that

$$m * L^\frown(a) = 1 \quad \text{if } a \in \bigcup A \cap y_i + \{1, 2, \dots, 2\lambda - 1\} + (-\delta, \delta).$$

Therefore by Theorem 2.1

$$\|m * L\| \geq C \sum_{i=1}^{N} |m * L^\frown(a_i)|/i$$

where $\{a_1 < a_2 < \cdots < a_N\}$ is the support of $m * L^\frown$. Thus we get that $\|m * L\| \geq C \sum_{i=1}^{n} \sum_{j=q_i+2}^{q_i+K_{n-i}} 1/j$ where $q_1 = 2K_{n-1} \geq |D_{n-1}|$ and for $i \geq 2$,

$$\begin{aligned} q_i &= \sum_{m=1}^{i-1} 6K_{n-m} + i - 1 + 4K_{n-i} \\ &\geq |D_{n-1}| + |C_{n-1}'| + \sum_{m=2}^{i-1} [|E_{n-m}| + |D_{n-m}| + |C_{n-m}'|] + |E_{n-i}| + |D_{n-i}|. \end{aligned}$$

Note that

$$\begin{aligned} q_i &\leq 6 \sum_{m=1}^{i-1} \prod_{j=m+1}^{i} (1 + p^{-1})^{r_{n-j}-1} p^{-1} 2^{-r_{n-j}} K_{n-i} + i - 1 + 4K_{n-i} \\ &\leq \left(6 \sum_{m=1}^{i-1} (2p)^{-i+m} + 5 \right) K_{n-i} \leq 11 K_{n-i}. \end{aligned}$$

Hence $\|m * L\| \geq C \sum_{i=1}^{n} \log(1 + K_{n-i}/q_i) \geq Cn \log(12/11)$. Because n was an arbitrary positive integer, this contradicts the boundedness of m. Q.E.D.

We do not know whether these results can be extended to other locally compact abelian groups with ordered dual. In view of [Ru, 8.1.5] it seems likely that the characterization of the complemented translation-invariant subspaces of $H^1(G) = \{f \in L^1(G) \colon \hat{f}(\gamma) = 0 \text{ for all } \gamma < 0\}$ can be reduced to the case of G compact by using our results. As far as we know only the cases $G = \mathbf{T}$ or $\mathbf{R}$ have been studied. In particular no characterization is known for $G = \mathbf{T}^n$, $n \geq 2$. (See [Ru, 8.1.7] for the definition of the order on $\mathbf{Z}^n$.)

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